Number Theory 21

10 August 2024 19:05

Q> $2^a + 2^b = c!$, and a, b, c are non-negative integers. Find the number of ordered tuple (a, b, c) satisfying above equation.

Ans:- WLOG Let $a \le b$

$2^a(\underline{1 + 2^{b-a}}) = c!$

$5 \mid (1 + 2^{b-a})$

$5 \nmid (2^a + 2^b)$

$1! = 1 \to \times$

$2! = 2 \to a = 0, b = 0, c = 2$

$3! = 6 \to (1, 2, 3), (2, 1, 3)$

$4! = 4 \to (3, 4, 4), (4, 3, 4)$

$\left.\begin{matrix} 5! = 0 \\ 6! = 0 \\ \vdots \quad = 0 \end{matrix}\right\} 0 \pmod 5$

$2^0 = 1$
$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 6$
$2^5 = 2$

$2^0 \equiv 1 \pmod 5$
$2^1 \equiv 2 \pmod 5$
$2^2 \equiv 4 \pmod 5$
$2^3 \equiv 3 \pmod 5$
$2^4 \equiv 1 \pmod 5$

Q> $P(x) = (x-1)(x^2-2)(x^3-3) \ldots \ldots (x^{11}-11)$. Find the coefficient of x^{60}.

Ans:- $\deg(P) = \frac{11 \times 12}{2} = 66 \longrightarrow$

		Coeffs
$(-1)(-2)(-3)$	$=$	-6
(-6)	$=$	-6
$(-5)(-1)$	$=$	5
$(-4)(-2)$	$=$	8
		1

Q> Find the sum of all natural numbers n such that $n^n + 1$ is prime and has at most 19 digits.

Ans:- Case I $n = \text{odd} \Rightarrow n^n + 1 = \text{even}$

$\Rightarrow \boxed{n = 1}$ is the only case as 2 is only even prime

$n^n < 10^{19}$ $\quad \approx n^n + 1 < 10^{19}$ (20 digits)

$\log_{10}(n^n + 1) < 19$

$\Rightarrow n \log_{10} n < 19$

$n = 2, 4, 6, 8, 10, 12, 14, 16, 18$

Case II $n = \boxed{\text{odd}} \times \text{even}$

$\hookrightarrow$ odd factor exist

$x^a + y^a = (x+y)(x^{a-1}y - x^{a-2}y^2 + \cdots)$

Case: n – $\boxed{\text{odd}} \times$ even

odd factor exist

$$n^{\text{odd} \times 2^c} + 1^{\text{odd}} = \underbrace{\left(n^{2^c} + 1\right)}_{>1}\left(\underbrace{n^{2^c(\text{odd}-1)} - n^{2^c(\text{odd}-2)}}_{>} + \cdots + 1\right)$$

take all such pairs + the last 1

$\Rightarrow > 1$

$\Rightarrow n^n + 1$ is not prime

Case III

$n = \text{even} = 2^c = 2, 4, 8, 16$

$2^{2^3} + 1$

$\boxed{n=2} \Rightarrow 2^2 + 1 = 5$

$\boxed{n=4} \Rightarrow 4^4 + 1 = 257$

$$n = 8 \Rightarrow 8^8 + 1 = 2^{24} + 1 = \underbrace{(2^8 + 1)}_{>1}\left(2^{8(3-1)} - \underbrace{2^{8(3-2)} + 2^{8(3-3)}}_{>1}\right)$$

not prime

$n = 16 \Rightarrow 16^{16} + 1 = 2^{2^6} + 1 \rightarrow$ it is a prime

$2^4 > 10$

$2^{64} = 2^{2^6} > 10^{16}$

$\log_{10} 2 > 0.3 \Rightarrow \log_{10} 16 > 1.2$

$2^3 < 10$

$16 \log_{10} 16 > 19.2 > 19$

$2^{57} < 10^{19}$

so 16 not in range

ans is $1 + 2 + 4 = 7$